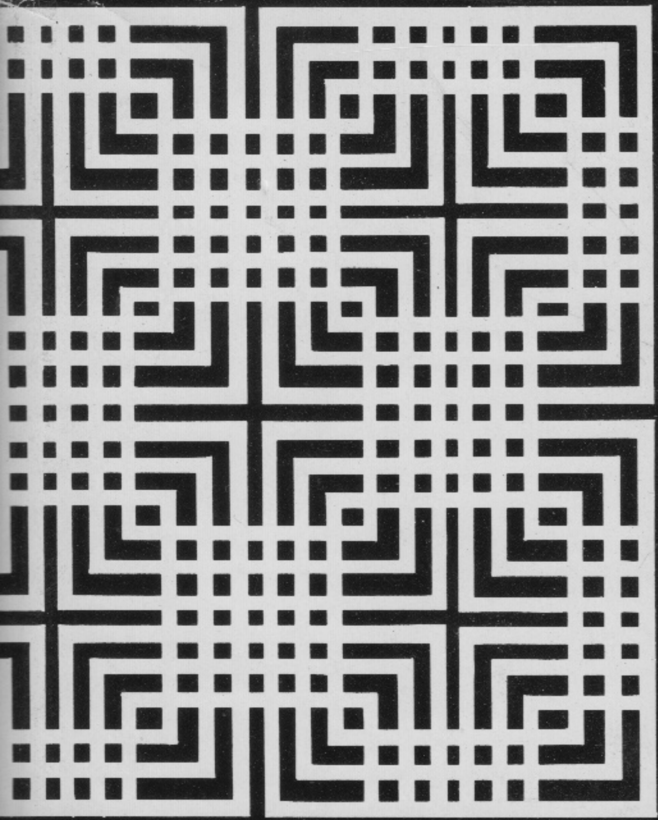
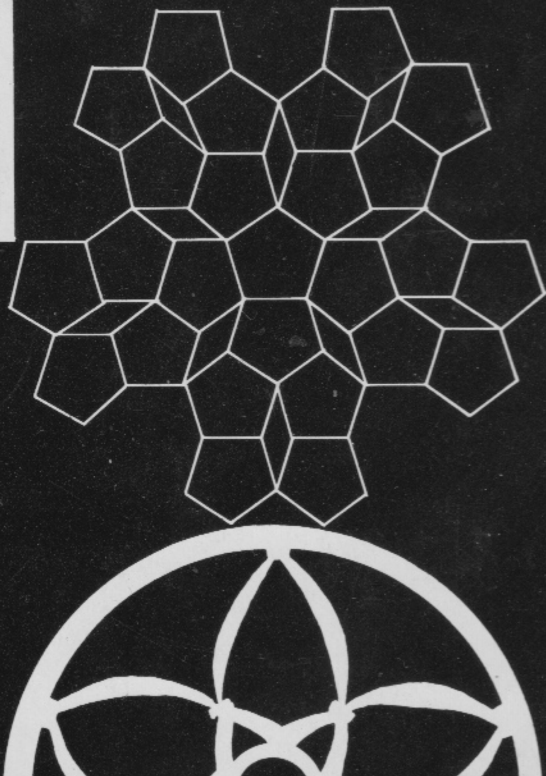
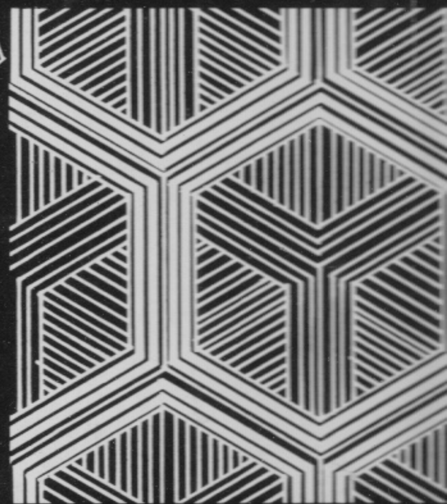


Handbook of Regular Patterns

An Introduction to Symmetry in Two Dimensions





Peter S. Stevens

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The Seventeen Plane Groups

Preface

A reading of George Birkhoff's *Aesthetic Measure* provided the inspiration for this work. Birkhoff's concerns for order and complexity in art led him to study different species of repetitive ornament including the seven band groups and the seventeen wallpaper groups. He provided two illustrations for each of those groups. Only two. I carefully studied his illustrations with the hope that I could generate variations of my own. I had only moderate success, however, and not until I reviewed the works of Weyl, Toth, and Buerger was I able to generate patterns easily. Even then I found it difficult to duplicate and imitate the more complex—and more interesting—designs in Owen Jones's *Grammar of Ornament*.

On the one hand there appeared to be an elegant mathematical system of pattern classification used by chemists and crystallographers, and on the other a wealth of repetitive designs in source books used by artists and graphic designers. Birkhoff and Weyl were comfortable in both worlds. After all, crystal sections and wallpaper designs obey the same structural rules—testimony to the harmony that underlies natural and man-made forms. Still, crystallographers and chemists have generally shown limited interest in the source books of artists, and artists know little of the underlying structural anatomy of repetitive groups.

This work attempts a synthesis of the two perspectives. It is an encyclopedia, a reference handbook, of repetitive designs organized in accord with established crystallographic notions of symmetry and symmetry operations.

On the crystallographic side I have adhered to the widely accepted system of classification and notation used in Henry and Lonsdale's *International Tables for X-Ray Crystallography.* For the band ornaments I have adopted a readily understood variation on that notation including the use of the symbols *tm* and *mt* as suggested by Martin Buerger, from whom I also borrowed the idea of using patterns of commas to picture the groups.

Despite its crystallographic backbone, however, the book is directed primarily to the practicing artist and designer. After an initial reading, and perhaps a few practice exercises, the graphic artist can use the chapter headings, running heads, and charts to locate any given structural arrangement. As a practical aid to design, the book provides extensive discussions of unit cells, fundamental regions, centered hexagons, and other generating units. In addition the appendix offers some of the mathematics that underlies repetitive structure. This volume, then, should enable the artist to produce original designs while having at his fingertips numerous examples of structurally similar designs from different cultures and historical periods.

The illustrative examples include designs from nature and architecture as well as ornamental patterns whose origins range from the first and second milleniums before Christ to the twentieth century. Several works are included of the late Dutch artist M. C. Escher, whose technique and use of color symmetry have been analyzed by Caroline Macgillavry in *Fantasy & Symmetry*. I have provided discussions of Escher's design symmetries in order to assist the reader in discovering how to generate additional Escher-like designs. For illustrative purposes I have also included several original designs, especially in the last chapter, which provides a visual recapitulation of the book.

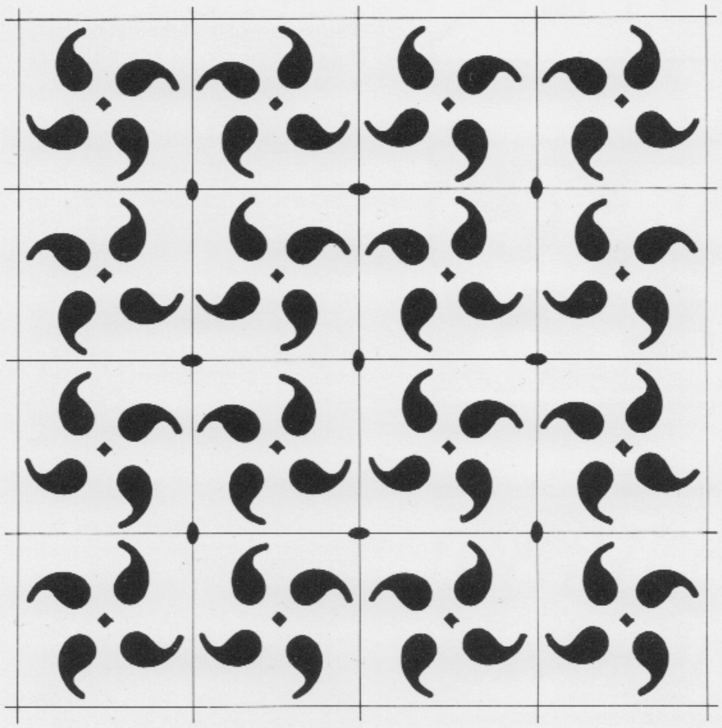
I am especially indebted to the M. C. Escher Foundation, Haags Gemeentemuseum, The Hague, Netherlands, for permission to reproduce the designs of M. C. Escher. I wish also to acknowledge the free use of many designs found in the Dover Pictorial Archive Series. Some of the books that make up that series can be found among the references.

Mollie Moran drafted most of the illustrations and I am deeply indebted to her. I would also like to thank Didi Stevens for her editorial review and Arthur Loeb for his reading of the manuscript and most helpful suggestions.

I
Symmetry Groups

See how various the forms and
how unvarying the principles.

Owen Jones



Here and elsewhere we shall not obtain the best insight into things until we actually see them growing from the beginning.

Aristotle

1
Basic Operations

The variety of ornamental pattern is extraordinary. All around us are patterns in textile designs, tiled floors, wallpapers, friezes, and carpets. Animal motifs and foliage patterns abound, but more numerous still are repetitions of abstract forms—circles, crescents, rectangles, and arrangements of stripes and lines. In the natural world too you find repetitive patterns—in the arms of the starfish, in a spiral galaxy, and in the arrangement of leaves on the branches of a tree. Every culture and historical period has produced unique forms and today's designers are busy turning out more.

Symmetry Groups

Mathematicians concerned with the theory of groups hold an interesting view of that variety. They ignore particularities and consider the ways that a motif repeats, the manner in which one part of a pattern relates to the others. The possibilities turn out to be strictly limited. Patterns that run in one direction, linear band ornaments, like those on the borders of wallpaper or the edges of crockery, come in only seven basic types—irrespective of whether the motifs are stylized scorpions along the edge of a Persian rug or leaves of honeysuckle on a Greek vase. Furthermore, you can make only seventeen two-dimensional patterns, patterns that cover a surface—like ceramic mosaics, tiled floors, and arrangements of brickwork.

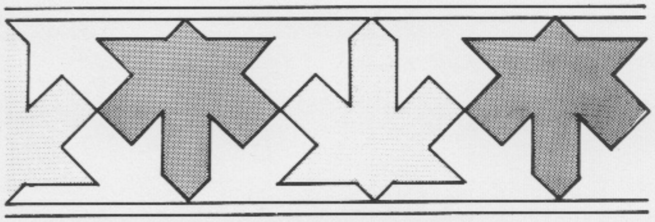
It was in 1935 that von Franz Steiger proved that only seventeen two-dimensional patterns exist. Unfortunately, Steiger's proof [8: pp. 235–249] makes use of abstract topological concepts that are not particularly useful to designers of patterns. A more useful approach is to catalog patterns in terms of their symmetries. Although repetitive designs have arbitrary features—after all, they may consist of clusters of triangles, birds, or flowers—their symmetries are fixed. Symmetries describe the ways that the arbitrary motifs can be manipulated.

Common Structures

Figures 1 and 2 illustrate the way in which the same structure can underlie different designs. Figure **1.1** shows part of a band of stylized leaves from a Persian ornament of the fifteenth century. This band illustrates one of the seven linear symmetry groups. Notice that the lightly shaded leaves that point up have precisely the same shape as the heavily shaded ones that point down. Consequently, if you turn the whole band upside-down, it has exactly the same appearance except that the colors reverse. Either way, the outlines of the leaves are precisely the same. Furthermore, each leaf is symmetrical in that its right-hand side is the mirror image of its left-hand side. Assuming that the band extends indefinitely to the right and to the left, you can see that the design as a whole is symmetrical. At the center of any leaf, the right-hand side of the entire band is the mirror image of the left-hand side. Later we will study rotations and mirror reflections in more detail and establish a precise meaning for the term symmetry group, but for now let us observe simply that the band ornament of figure 1 has the same appearance whether upright or upside-down and that the left-hand side is the mirror image of the right-hand side.

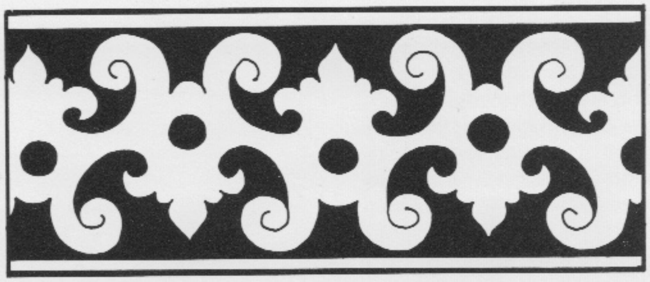
Can we find other examples of the same structure? Consider the band ornaments in figure **1.2**. Frame (*a*) shows a band from Brazil dating from the fourth or fifth century; frame (*b*), a strap ornament from Elizabethan England; frame (*c*), an ornamental frieze from ancient Greece; and frame (*d*), a contemporary design from the United States. Although these ornaments differ from the Persian ornament of figure 1.1, they have exactly the same underlying structure. Each band looks the same whether upright or upside-down, and each band can be divided so that one half is the mirror image of the other half. Whether Persian, Brazilian, English, Greek, or from the United States, whether abstract or floral, whether ancient or modern, all these designs follow the same plan; all have the same anatomy; all belong to the same species of ornament or symmetry group.

Knowledge of symmetry groups can be a great aid to designers. Suppose, for example, that you manufacture patterned concrete blocks. What varieties of design can you offer your customers? How many different walls with repeating patterns can you make from different arrangements of a single motif? If you arrange the blocks in accord with the rules, you can quickly obtain the answers. You will find that any asymmetrical motif can be stacked with itself to create seven linear bands and seventeen planar patterns.

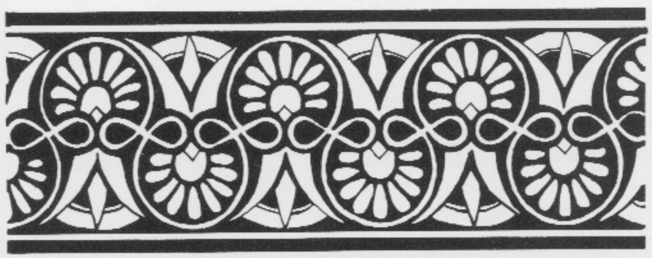


1.1

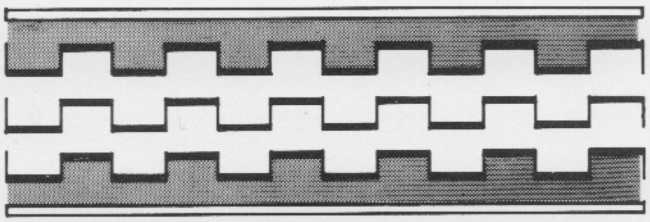
1.2a



1.2b



1.2c



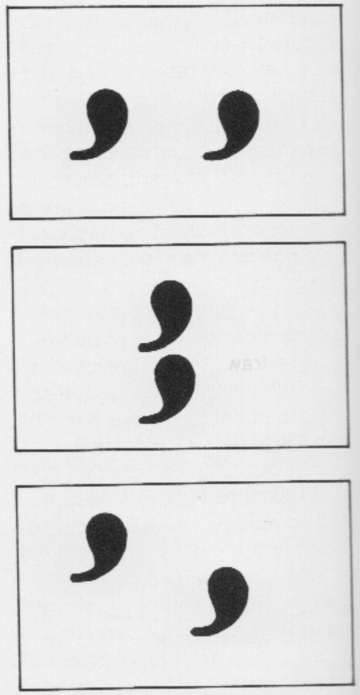
1.2d

Four Symmetry Operations
You will gain a good start toward understanding the structure of regular patterns when you realize that a two-dimensional motif can repeat in only four different ways. These four types of repetition, or symmetry operations as they are called, are worth studying because combinations of them produce all of the different symmetry groups. The four operations are (1) translation, in which the motif moves up or down, left or right, or diagonally while keeping the same orientation; (2) rotation, in which the motif turns; (3) reflection, in which the motif reflects as in a mirror; and (4) glide reflection, in which the motif both translates and reflects.

Repetition of the asymmetric comma shown in figure **1.3** illustrates these four operations in an elegant fashion. For the ancient Persians the comma—the familiar paisley pattern as well as our common punctuation mark—was a venerated motif. Some scholars consider the mark a stylized flame and trace its history through Zoroastrian cults to the primitive worship of fire. Others prefer the tale that an Iranian artist's young son dipped his hand into his father's pot of paint and imprinted a piece of linen with the side of his half-curled fist.


1.3

Translation
The commas in figure **1.4** show three examples of the first symmetry operation, translation. The frames of the figure show the commas translated horizontally, vertically, and diagonally. You can imagine that the Iranian boy stood opposite you as you look at the page and, working with his left hand, kept his wrist stiff as he made his marks. Each comma in a framed pair is a translation of the other. This idea is important because every operation is a two-way street. Every operation shifts the second image into the first image as well as the first image into the second. If you view all three frames as a totality, commas in any pair are each other's translation, irrespective of whether they reside in the same frame.

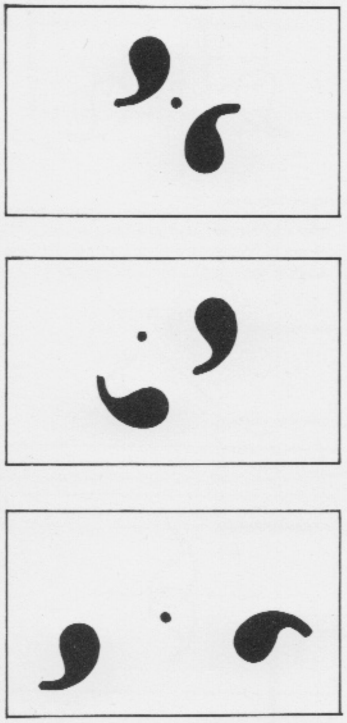

1.4

Rotation

Figure **1.5** illustrates the second symmetry operation with three sets of commas. In these examples the boy turned his fist between the first and second print of each pair. It is important to realize that no translations are involved; only rotations. If you rotate one comma of a pair around the point marked in the frame, you can superimpose it exactly on the other comma. To be certain, redraw one of the commas in the first frame on a piece of tracing paper and, with your pencil, pin the paper to the dot that marks the center of rotation; you can then turn the paper so that the comma you have drawn coincides with the other. Do the same for the commas in the other frames. Now if you consider the three frames together, you find that any comma in any frame is a rotation of any of the others. Try to locate, for example, the center of rotation that relates the lower comma in the top frame to the upper comma in the middle frame. See if you can locate similar centers of rotation for other pairs.

As you study figure 1.5 you will discover that distant centers of rotation, or rotocenters as they are commonly called, produce apparent translations. As a case in point, note the apparent translation between the upper commas in the top and middle frames. Those commas are related by rotation through a distant rotocenter.

Figure **1.6** shows the idea with greater clarity. The dot in the top frame is the center of rotation for all three pairs of commas. Each successive pair, at increasing distance from the rotocenter, shows less twist and more sweep. Consequently we can see that pure translation is but a special case of rotation—rotation through an infinitely small angle or, what amounts to the same thing, rotation around a center that lies infinitely far away.



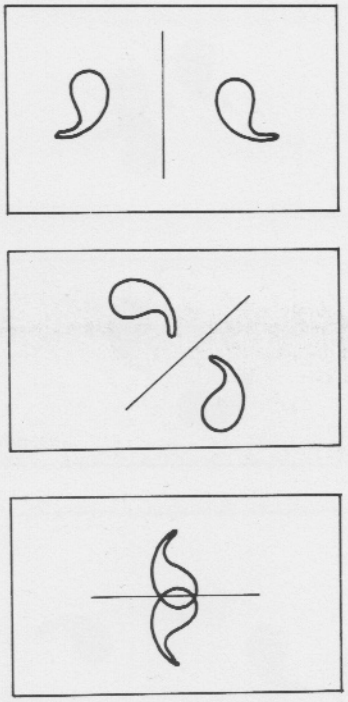
1.5

1.6

Through the Looking Glass
How did the boy make the marks in the frames of figure **1.7**? He used first one hand and then the other. The commas in each pair are similar and yet as intrinsically different as a left and a right hand, for as with opposite hands, one comma is the mirror image of the other.

If you place a small hand-mirror along the centered line in each frame, the comma in front of the mirror will reflect into the mirror to adopt the position of the other comma. The mirror lines in the figure actually represent ideal mirrors, mirrors without thickness that reflect perfectly from both sides. Again, then, you must look at a given pattern in its entirety, remembering that a symmetry operation cuts both ways: what it does to one image it does in reverse to the complementary image. When the mirror line cuts a comma, as in the third frame, both pieces reflect so that a complete comma and its reflected image lie on top of one another. You should note in passing that in this book mirror lines are always represented by solid lines.

So reflection, the third symmetry operation, differs dramatically from translation and rotation. A motif that is translated or rotated glides across the plane surface, and its image at the start can be directly superimposed on its image at the finish or anywhere along the way. The operation of reflection lacks that continuity. The motif either reflects or not; there is no in-between. Furthermore, the motif does not stay on the surface of the plane throughout the operation. It jumps out of the plane and flips over on its back: it reverses. Consequently, to obtain a reflection without using a mirror, you need only view from the reverse side a motif drawn on the front of a piece of tracing paper. Looking through the paper automatically reverses the image.

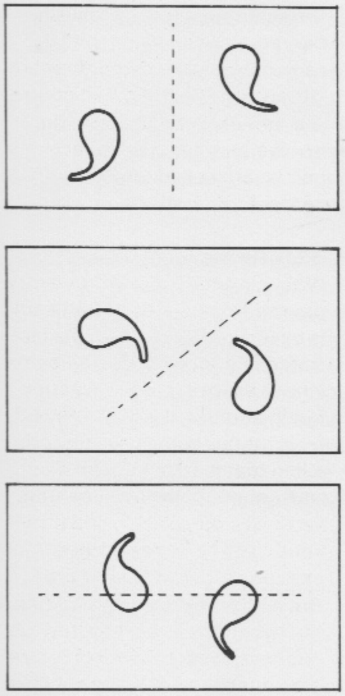


1.7

At this point you might wish to tease your brain with the question of how you could reverse your right hand to get a left hand. The answer is that you can look into a mirror. Otherwise, in order to transform a right hand into a left, you must lift it out of the third dimension into the fourth, turn it over, and bring it back—a bit difficult. Apparently though, you do something similar when you transform your right-hand glove into a left-hand glove by turning it inside out.

Because reversal involves a flip rather than a glide, mirror reflection is called an indirect or "improper" operation. In contrast rotation and translation, which involve only simple displacements, are called direct or "proper" operations.

Glide Reflection
What about the fourth symmetry operation? It is also indirect and is called glide reflection. As illustrated in figure **1.8**, the image reverses as with mirror reflection, but in addition it glides. Instead of a mirror line, we have a glide line, a line that marks the path of a translation. This glide line runs vertically, diagonally, and horizontally in the three frames, much like the mirror lines in figure 1.7. But as you compare the frames of the two figures, you see that the commas in each frame of figure 1.8 translate as well as reflect. In figure 1.8 you may be able to locate the unmarked glide line between the comma on the right in the top frame and the comma on the left in the middle frame. Perhaps you can also locate unmarked glide lines between the other pairs of commas. Again in passing, you should note that dotted lines like those shown in



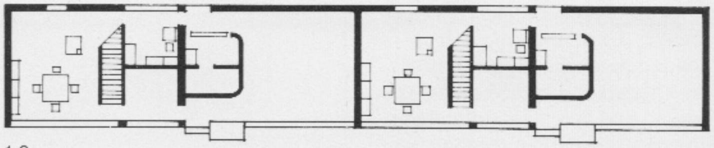
1.8

each frame of figure 1.8 are used throughout this book to represent lines of glide reflection.

Just as translation is a special case of rotation, mirror reflection is a special case of glide reflection. Mirror reflection is glide reflection with zero glide. Thus the four symmetry operations—translation, rotation, reflection, and glide reflection—reduce to only two—rotation and glide reflection.

Architectural Examples

Figure **1.9** is especially revealing. It illustrates the use of all four operations in the development of architectural plans for housing at Pessac, near Bordeaux. The plans were conceived by the Swiss architect, Le Corbusier. In frame (a) he translated a basic plan to make two identical units side-by-side. In frame (b) he repeated a unit after rotating it 180° so that two units interlock. In frame (c) he mirror-reflected the plans and in frame (d) he made a glide reflection.



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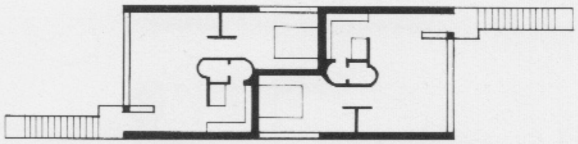

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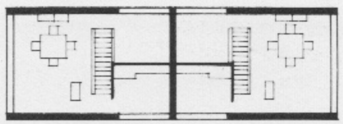
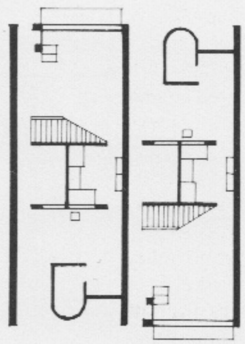
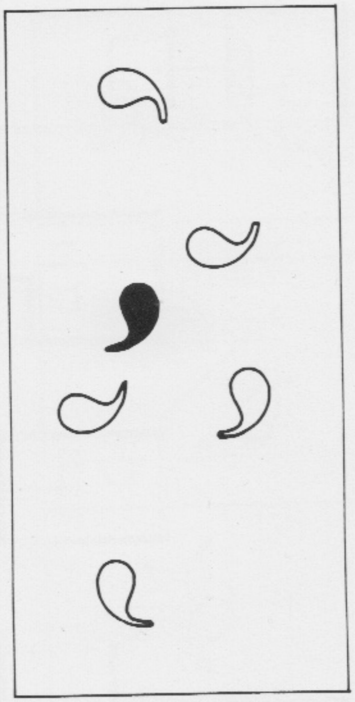


1.9c



1.9d

By way of review, see if you can identify the relation of each unshaded comma in figure **1.10** to the shaded one in the center. You should find two glide reflections and one translation, rotation, and mirror reflection. Which operations are direct? Which indirect?



1.10

Exercises

1. What operations describe the relation between the two triangles in each frame of figure **1.11**? What special characteristic of the triangles in the last frame leads you to name two different operations?

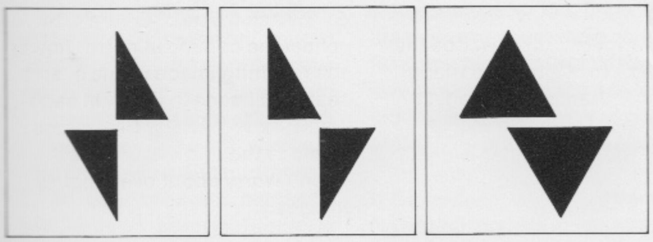
2. Sprinkle cardboard cutouts of an asymmetric motif on a surface and describe the relations among them. How are those that flip over related to those that do not?

3. In figure **1.12**, reflect the comma in mirror line *a*. Then reflect the resulting image in line *b*. How does the final image relate to the original comma? Relative to the distance between mirror lines, how far is the final image from the original? Would the results be the same for every motif reflected in parallel mirrors?

4. In figure **1.13**, reflect the comma in mirror line *a*. Then reflect the resulting image in mirror line *b*. How is the final image related to the original comma? If the angle between the two lines is 45°, what is the angle between the final image and the original? What would be the angle between the final and original image if the angle between the lines was 90°?

5. In each frame of Figure **1.14**, reflect the comma in mirror line *a*, the resulting image in line *b*, and that next image in line *c*. In each frame, how does the final image relate to the original comma? (Don't worry about overlapping images.)

6. Show that a translation results from successive reflections in two parallel mirrors, a rotation results from reflection in two mirrors that intersect, and a glide reflection results from successive reflections in two parallel and one perpendicular mirror. (This exercise should provide a check for the others.)



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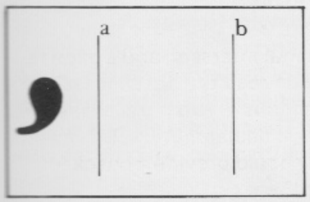

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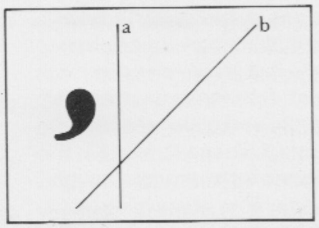

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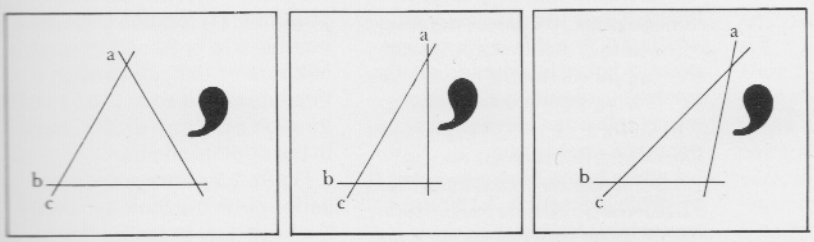


1.14

We are no longer preoccupied with mere facts, but with the relations which the facts have for one another—with the whole which they form and fill, not with the parts.

Jacob Bronowski

2
How Operations Generate Themselves

With the aid of displacements and reversals of commas, we have examined the four symmetry operations that produce repetition. We are now in a position to tackle the meaning of symmetry.

Symmetry

If the comma is an asymmetric figure, what is a symmetric one? The answer is that a symmetric figure has repetitive parts. And how might the parts of a figure repeat? Just as we have discovered: by translation, rotation, reflection, and glide reflection.

Letters A, B, and C, for instance, are symmetrical through reflection. They enjoy bilateral symmetry in which one half is the mirror image of the other. A mirror through the center of these letters leaves their appearance unaltered. And this is the critical point. A figure is symmetrical if a symmetry operation such as a mirror reflection leaves its appearance unchanged.

Letters N and Z are symmetrical by virtue of rotation. A 180°-turn leaves their appearance unchanged because one half of the letter is exactly like the other. You may find the symmetry of N and Z surprising since we commonly use the word symmetrical to mean bilaterally symmetrical. We would all agree, for example, that the letter A is symmetrical. Although the letters N and Z are not symmetrical in the manner of A, they are also symmetrical since their parts repeat by rotation. Similarly, the word "bud" is symmetrical because of mirror reflection, the word "pod" is symmetrical under rotation, and the word "dodo" is symmetrical by virtue of translation.

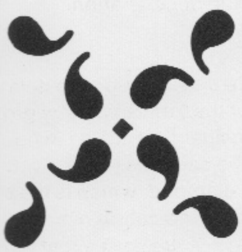
Symmetry Groups

We are closing in on the definition of the term symmetry group.

Consider the pattern in figure **2.1**. The center is marked with a small diamond, which is the conventional indication of a fourfold rotation. You can see that four operations (rotations) displace the pattern to four equivalent positions: (1) rotation through a quarter-turn or 90°; (2) through a half-turn or 180°; (3) through a three-quarter turn or 270°; and (4) through a full turn or 360°, back to the original position.

Figure **2.2** portrays another pattern with the same symmetry. It is important to realize that although each arm of figure 2.2 contains two commas related by a mirror, the individual mirrors do not reflect the entire pattern into

2.1



2.2

equivalent positions. The pattern then—as a whole—has only fourfold rotational symmetry. Consequently, figures 2.1 and 2.2 belong to the same symmetry group.

A Definition

Now then, the definition of the term symmetry group. A symmetry group is a collection of symmetry operations that together share three characteristics: (1) each operation can be followed by a second operation to produce a third operation that itself is a member of the group, (2) each operation can be undone by another operation, that is to say, for each operation there exists an inverse operation, and (3) the position of the pattern after an operation can be the same as before the operation, that is, there exists an identity operation which leaves the figure unchanged.

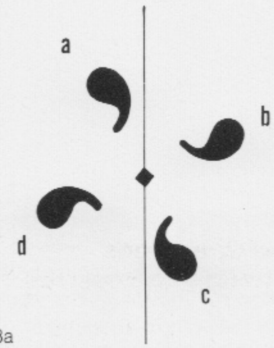
You can see that the definition is very abstract. Perhaps though, you can see how the operations that rotate figures 2.1 and 2.2 form a symmetry group. Here is an enumeration that accords with the definition. (1) The 90° rotation can be followed by a 180° rotation to produce a 270° rotation which itself is a member of the group. (2) A 90° rotation can be undone by a 90° rotation in the opposite direction. Any of the other rotations can similarly be followed by inverse rotations. (3) A full turn of 360° brings the pattern back to where it began. Any rotation that is a multiple of 360° also leaves the pattern in the same position.

Generating Groups

Where do we go from here? At this point we can see how the operations feed upon themselves to produce still larger groups. As an example, let us add another operation to the fourfold center. Let us pass a mirror directly through its heart and study the results step-by-step.

Figure **2.3**a shows a mirror through a fourfold center, with each of the four arms identified. As indicated in figure 2.3b, arms *a*, *b*, *c*, and *d* reflect to produce arms *a'*, *b'*, *c'*, and *d'*. The result of all those reflections appears in figure 2.3c. Now you find that the pattern contains four intersecting mirrors. A single mirror through a fourfold rotocenter produces automatically four intersecting mirrors.

All the reflections and the rotations together form a symmetry group because they obey the following conditions: they produce



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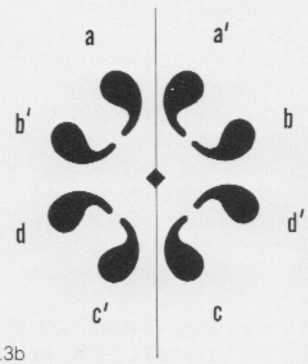

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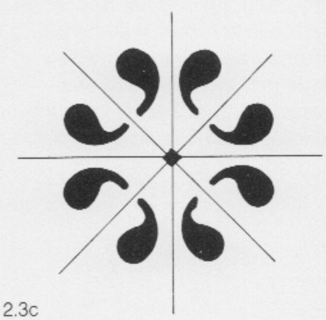


2.3c

additional reflections and rotations, they can be undone by inverse operations, and some combinations of them leave the position of the pattern unchanged. The entire symmetry group contains eight operations: four reflections and four rotations. All these operations leave the appearance of the pattern unaltered.

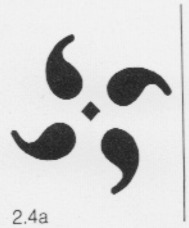
As another illustration of the manner in which symmetry groups come into being, consider placing the mirror line next to a fourfold design instead of through its heart. The arrangement is depicted in figure **2.4**a.

Taking the results step-by-step, you find that the mirror reflects the rotating pattern to produce the double image shown in figure 2.4b. You can see how the two rotocenters interrelate like a pair of oppositely rotating paddle wheels. But this is only part of the story, for if the original rotocenter continues to rotate, it shifts the mirror along with the reflected image into the four positions shown in Figure 2.4c. It thus produces four mirrors. At this point, the action continues because each mirror reflects and each rotocenter rotates, and all together they generate, in the twinkling of the eye, an endless array of images in all directions across the plane. A portion of the infinite pattern appears in figure **2.5.**

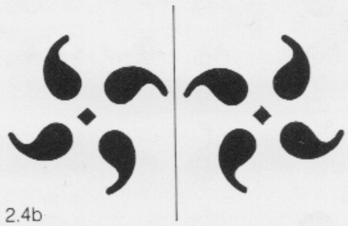
What are the operations in figure 2.5?

First there are fourfold rotocenters like the motif of figure 2.1, except that in the infinite pattern the whole pattern rotates. In other words, the rotating motif consists now of more than four commas; it includes the whole two-dimensional plane with all the fourfold centers and all the mirror lines. To verify that the entire pattern repeats with each 90° rotation, draw the pattern on tracing paper and rotate it around each rotocenter. You will see the entire pattern repeat again and again. In addition to clockwise rotations you find mirror-image counterclockwise rotations. Both types of rotocenter act on the entire plane.

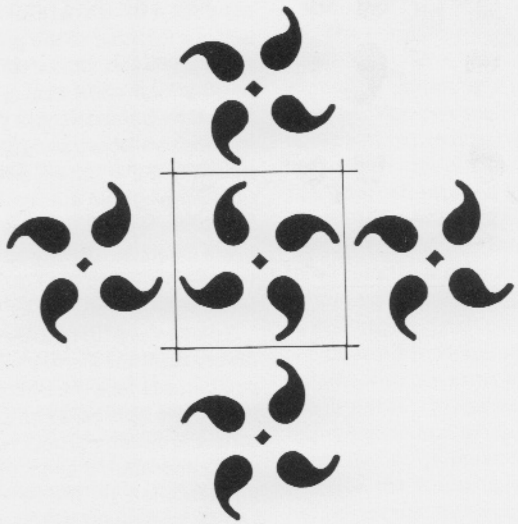
Next you find that the infinite pattern has generated two different sets of parallel mirror lines that intersect perpendicularly. Every mirror reflects the entire infinite pattern into itself.



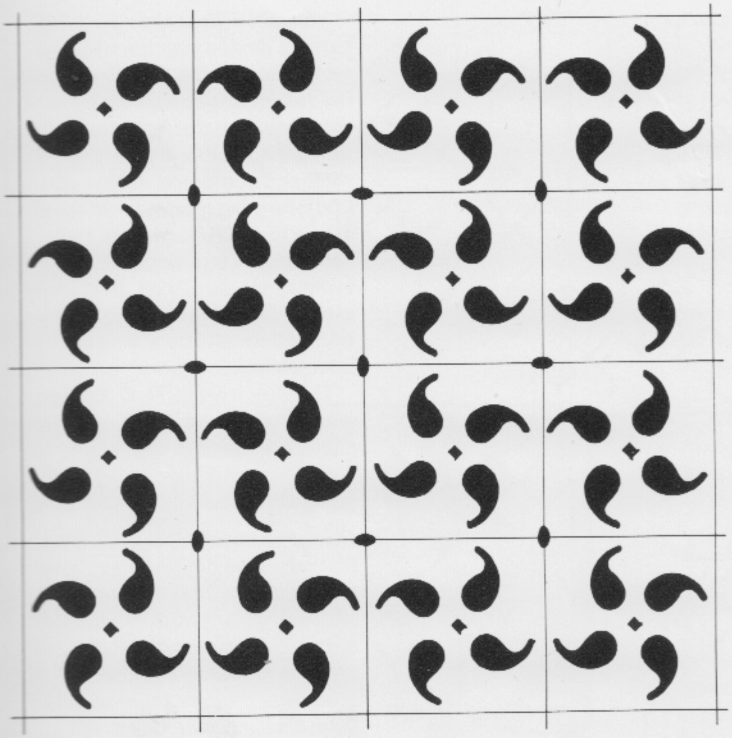
2.4a



2.4b



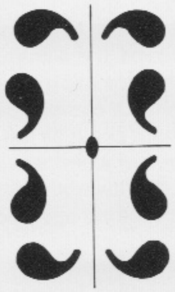
2.4c



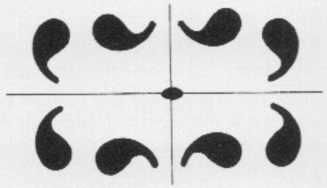
2.5

Last, you discover twofold rotocenters that lie on each intersection of the mirror lines. In accord with convention they are marked with an oval. Figure **2.6** shows a small portion of the twofold center in its two different orientations. Each twofold center contains the same symmetry operations as the letter H: a vertical reflection, a horizontal reflection, and a 180° rotation.

As with the mirrors and the fourfold rotations, remember that each twofold center acts not only on the eight commas shown in figure 2.6 but on all the commas in the infinite pattern. In other words the entire pattern forms a twofold rotocenter. Even more to the point, the entire pattern forms an infinite number of twofold rotocenters. Some have the orientation shown in figure 2.6a, the others have the orientation shown in figure 2.6b. In addition, as already described, the pattern forms an infinite number of left-handed and right-handed fourfold centers—plus an infinite number of perpendicularly intersecting mirror lines. All these elements are interrelated and perfectly spaced to repeat and regenerate endlessly. All this interactive creation and re-creation flashes into existence when a fourfold rotation mates with a mirror line.



2.6a



2.6b

Figure 2.5, then, illustrates one of the seventeen two-dimensional plane groups. It is a uniquely interactive association of symmetry operations. Of course, the repetitive element could be the letter P, a flower, or a bird, as well as a comma. The invariant elements are the structural operations.

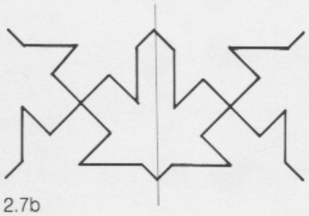
A Further Example

For comparison, let us look at another example of the same symmetry group. Figure **2.7** shows (a) a fourfold center, (b) the same center reflected once, and (c) the same center reflected four times. Further rotations and reflections produce the design of figure **2.8** which, when colored, results in the leaf pattern of figure **2.9.**

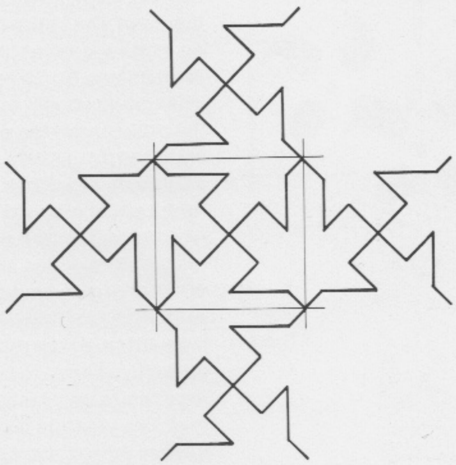
This pattern occurred as a painted decoration in a house in Cairo in the fifteenth century. It consists of only a single motif expressed in three different colors. A linear band of the pattern was depicted in figure 1.1

The interesting point is that the leaf form in figure 2.9 performs exactly the same maneuvers as the comma in figure 2.5—clockwise and counterclockwise fourfold rotations, mirror reflections, and twofold rotations. Figures 2.5 and 2.9 express exactly the same thought, but in different languages.

2.7a



2.7b

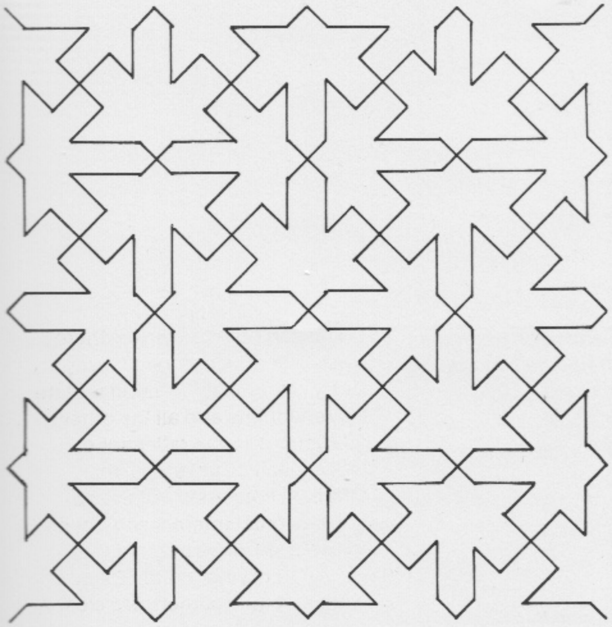


2.7c

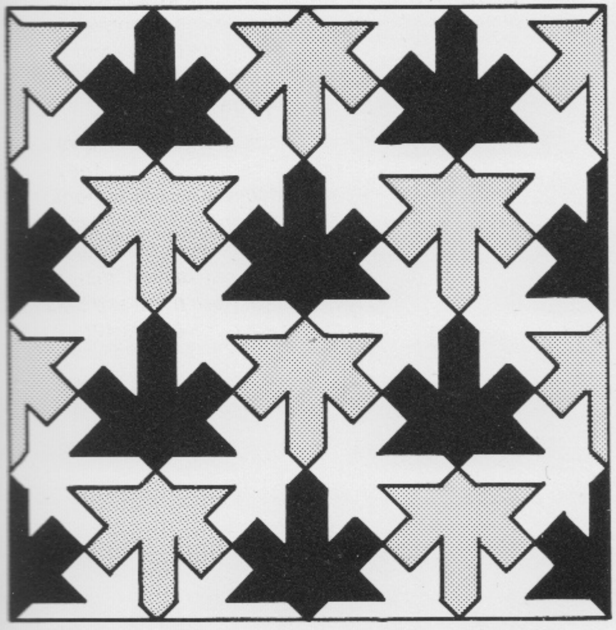
You can now sense why there exists a limited number of symmetry groups: the rotations rotate the reflections and all the other rotations, and the reflections reflect each other and all the rotations combined. All the operations must interact with themselves to produce more of the same sort of interaction. Each symmetry group, then, is a closed system of self-generating operations where all operations are interrelated.

The Dutch artist, M. C. Escher, who created a wealth of novel designs, grasped the beauty and lawfulness of these self-generating operations:

There is something in such laws that takes the breath away. They are not discoveries or inventions of the human mind, but exist independently of us. In a moment of clarity, one can at most discover that they are there and take them into account. [52: p. 40]



2.8



2.9

Exercises

1. What symmetry operation leaves the appearance of the comma in figure 1.3 unchanged?

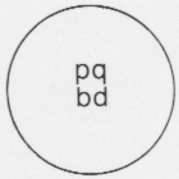
2. What two symmetry operations make up the symmetry group described by the commas in the first frame of figure 1.4?

3. How many symmetry operations make up the group described by the commas in the first frame of figure 1.5?

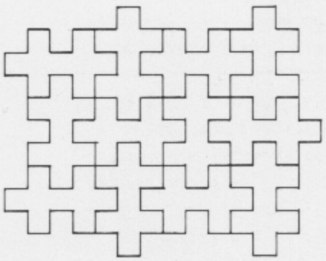
4. What operations leave unchanged the appearance of the letters in figure **2.10**?

5. If instead of adding a mirror to the side of the fourfold rotocenter shown in figure 2.4a, you add it to the side of the rotocenter shown in figure 2.3c, would the resulting infinite pattern contain the same operations as the pattern of figure 2.5? Do you think the two infinite patterns would belong to the same symmetry group?

6. What operations exist in figure **2.11**? Does the pattern belong to the symmetry group displayed in figure 2.5?



2.10



2.11